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Applications of partial supersymmetry

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Abstract
I examine quantum mechanical Hamiltonians with partial supersymmetry, and explore two main applications. First, I analyse a theory with a logarithmic spectrum, and show how to use partial supersymmetry to reveal the underlying structure of this theory. This method reveals an intriguing equivalence between two formulations of this theory, one of which is one dimensional, and the other of which is infinite-dimensional. Second, I demonstrate the use of partial supersymmetry as a tool to obtain the asymptotic energy levels in non-relativistic quantum mechanics in an exceptionally easy way. In the end, I discuss possible extensions of this work, including the possible connections between partial supersymmetry and renormalization group arguments.

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1. Introduction

Supersymmetry is, by now, a familiar mathematical construct for physicists as well as mathematicians. It has been invoked not simply as a possible phenomenological symmetry of nature, but as a tool with which one can analyse generic, even non-supersymmetric, physical theories, as well as derive mathematical results. Examples of non-supersymmetric applications of supersymmetry range from the use of shape invariance to solve exactly soluble quantum Hamiltonians [1], the analysis of monopoles and other topological field configurations [2] and the calculation of gluon and graviton scattering amplitudes [3].

One of the hallmarks of supersymmetry is that it produces structures that afford exact control over at least some aspects of a physical theory, which in turn leads to the frequent effectiveness of supersymmetry as an analytical tool when a model can in some fashion be associated with a supersymmetric theory. Given the extensive usefulness of supersymmetry, any generalizations of supersymmetry are also potentially of great interest. A few years ago, in a work on arithmetic quantum theories [4], I introduced a notion of partial supersymmetry, in which some, but not all, of the operators and states of a theory were paired in supersymmetric fashion. In that paper, I established a number of uses for partial supersymmetry, including the ability to interpolate continuously among a class of parafermionic theory of different orders, and the identification of equivalences among certain arithmetic quantum theories that appeared to be distinct, yet describe the same physics.

In the present paper, I return to the subject of partial supersymmetry, and demonstrate that it is also a valuable tool in analysing quantum mechanics. Here, the intention is not, as it was in [4], to obtain new results with partial supersymmetry; rather, this paper is in the spirit of the introduction of shape invariance, which was important not because it showed how to solve models that could not be solved any other way, but because it made the actual solution very easy to execute, and revealed the structure underlying exact solvability.

First, I will state briefly what is meant by partial supersymmetry. Then I will use partial supersymmetry to study a theory with a logarithmic spectrum, a spectrum which is of interest for its role connecting quantum mechanics both to number theory [5] and to string theory [6]. Partial supersymmetry will reveal the underlying structure of this theory. Moving from a specific application to a more general one, I will use partial supersymmetry as a tool to extract the asymptotic energy levels of quantum mechanical systems. While there are already techniques to obtain asymptotic energy levels, the methods described in this section are significant both for their ease of use—indeed, there is probably no easier way to obtain the asymptotic spacing of energy levels in non-relativistic quantum mechanics—and for the way in which they reveal the symmetry structure underlying the fact that the asymptotic behaviour can be exactly calculated. Finally, I discuss some possible interpretations of this symmetry, and consider additional applications.

2. Partial supersymmetry

In this paper, we will for the most part work with one-dimensional non-relativistic quantum mechanics in the Schrödinger picture.

Consider a Hamiltonian H_0. The spectrum of this theory will consist of a set of states with energies $E_0 < E_1 < E_2 < E_3 < \cdots$. For simplicity, we will consider the case with a purely discrete spectrum.

Now what happens if we extend this to the supersymmetric case?[1] In this case, the energies are bounded from below by zero. The states of positive energy come in degenerate pairs, while there may also be a supersymmetry singlet zero energy state. Thus the supersymmetric scenario leads to two sectors, either both with the same spectrum, or the situation in which one sector has states with energy $E_0 < E_1 < E_2 < E_3 < \cdots$, and the other sector consists of states of energy $E_1 < E_2 < E_3 < \cdots$. Needless to say, the supersymmetry algebra implies results beyond simply level degeneracies, but we will not focus on those here.

Now suppose the spectrum has a different structure. Consider the case that there are two sectors, one with energies $E_0 < E_1 < E_2 < E_3 < \cdots$, and the other with energies $E_1 < E_3 < E_5 < \cdots$. In this case, every state in the second sector has a partner from the first sector with degenerate energy, but only every other state in the original sector has such a partner in the second sector. This situation, in which all the states in one sector have partners, but only some of the states in the other sector do, and in which there is a regular pattern to which states do and do not have partners[2], is the situation we shall term *partial supersymmetry*. In fact, the underlying structure should have implications beyond degeneracies, e.g., that some

[1] While strictly speaking, one might think of supersymmetry as only the requirement that H can be written as the square of a Hermitian operator, in order for supersymmetric quantum mechanics to make contact with conventional supersymmetric field theory, we also require a Z_2 grading analogous to $(-1)^F$.

[2] Without some regularity, after all, one should not expect an underlying structure.

operators come in pairs, and others are singlets, as discussed in [4], but for the purposes of this paper, we will focus on the spectrum.

In the remainder of this paper, we will examine some situations in which partial supersymmetry can be found and utilized.

3. The logarithmic spectrum

Suppose we have a quantum mechanical Hamiltonian

$$H_L = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_L(x) \tag{3.1}$$

which has the property that its energy spectrum is purely logarithmic,

$$E_n = \epsilon_0 \ln(n) \qquad n = 1, 2, 3, \ldots. \tag{3.2}$$

Such a theory has proved to be of interest in [4–6] as a toy model of some features of string theory, in particular the appearance of a Hagedorn temperature [7]. This theory and the related ones, in which the spectrum is still logarithmic but there are degeneracies at various levels, have also been useful as tools to draw connections between quantum mechanics and arithmetic number theory, invoking, for example, the observation that the physical quantum partition function for such a theory is the number theoretic Dirichlet series of the degeneracy function (with the non-degenerate case yielding the Riemann zeta function). As a result, understanding the Hamiltonian leading to such a spectrum is something that has led to much interest, and any insight we can get into the structure of such a theory is valuable[3].

Here, our focus is not on the many uses of the logarithmic spectrum, but rather on the structure of the theory that has a logarithmic spectrum without degeneracies, and the way in which partial supersymmetry can reveal that structure. In the present section, we will not discuss the form of the potential $V_L(x)$, but we will come back to that question later in this paper.

Return, then, to the consideration of the Hamiltonian H_L. Suppose we now define a new Hamiltonian, $\tilde{H}_L = H_L + \epsilon_0 \ln(2)$. On the one hand, adding a constant has no effect other than to shift the energies. On the other hand, in the case at hand, something interesting arises.

The spectrum of $\tilde{H}_L$ consists of the energies

$$\tilde{E}_n = \epsilon_0 \ln(n) + \epsilon_0 \ln(2) = \epsilon_0 \ln(2n) \qquad n = 1, 2, 3, \ldots. \tag{3.3}$$

One notes that these levels are degenerate with the alternate levels of the original Hamiltonian H_L. Consequently, the matrix Hamiltonian $\mathrm{diag}(H_L, \tilde{H}_L)$ that combines these two systems has two sectors, one consisting of the spectrum of H_L and the other consisting of the spectrum $\tilde{H}_L$, and is therefore partially supersymmetric.

We can write the Hamiltonian for this combined system in an alternate way that is quite instructive. Introducing fermionic creation and annihilation operators $f_2^\dagger$ and f_2, respectively, which satisfy the standard anticommutation relations

$$\{f_2, f_2\} = 0 = \left\{f_2^\dagger, f_2^\dagger\right\} \qquad \left\{f_2^\dagger, f_2\right\} = 1 \tag{3.4}$$

we now define the combination Hamiltonian

$$H_C = H_L + \epsilon_0 \ln(2) f_2^\dagger f_2. \tag{3.5}$$

This theory has two sectors, one with fermion number zero, which has the logarithmic energy spectrum of H_L, and one with fermion number +1, which has the even logarithmic energy spectrum of $\tilde{H}_L$.

[3] The interested reader is encouraged to examine the resources on number theory and physics at `http://www.maths.ex.ac.uk/~mwatkins/zeta/physics.htm`.

With this result in hand, we now begin to construct a Hamiltonian H_E which has the same spectrum as, and is thus thermodynamically equivalent to, the original Hamiltonian H_L. Since H_C exhibits a partial supersymmetry invariance, we expect that within H_L, there will be energy levels equal to those generated by a bosonic term partner to the fermionic addition in (3.5). Hence we anticipate that H_E will contain a term of the form $\epsilon_0 \ln(2) b_2^\dagger b_2$, which we shall denote with the notation

$$H_E \supset \epsilon_0 \ln(2) b_2^\dagger b_2 \tag{3.6}$$

where we have introduced the bosonic creation and annihilation operators satisfying the standard algebra

$$[b_2, b_2] = 0 = \left[b_2^\dagger, b_2^\dagger\right] \qquad \left[b_2^\dagger, b_2\right] = 1. \tag{3.7}$$

Now there was nothing special about adding energy $\epsilon_0 \ln(2)$ to the original Hamiltonian; one could instead have added $\epsilon_0 \ln(3)$, for example, and then pairing this new system with the original H_L, we would have a system in which every third state of the original Hamiltonian has a degenerate partner. The associated partial supersymmetry in this case would then indicate that

$$H_E \supset \epsilon_0 \ln(3) b_3^\dagger b_3. \tag{3.8}$$

Likewise, we would expect that for any other integer

$$H_E \supset \epsilon_0 \ln(j) b_j^\dagger b_j. \tag{3.9}$$

However, this is too much. For example, once one has included a term $\epsilon_0 \ln(2) b_2^\dagger b_2$ in H_E, this already produces a state with energy $\epsilon_0 \ln(4)$, and so it is overcounting also to include in H_E a term of the form $\epsilon_0 \ln(4) b_4^\dagger b_4$. Likewise, once $\epsilon_0 \ln(2) b_2^\dagger b_2$ and $\epsilon_0 \ln(3) b_3^\dagger b_3$ are both present in H_E, this already produces a state with energy $\epsilon_0 \ln(6)$, and so it is overcounting also to include in H_E a term of the form $\epsilon_0 \ln(6) b_6^\dagger b_6$.

This process described above of determining which bosonic operators to include and which to exclude is readily recognized as an implementation of the Sieve of Eratosthenes [8], and so the only terms that should remain in H_L are those associated with prime integers. Consequently, we conclude that we can write the Hamiltonian H_E as

$$H_E = \sum_{k=1}^{\infty} \epsilon_0 \ln(p_k) b_k^\dagger b_k \tag{3.10}$$

where p_k is the kth prime, and we have re-labelled the operators so that $b_k^\dagger$ and b_k are associated with the integer p_k rather than k.

While it is clear by direct calculation that the right-hand side of (3.10) has a logarithmic spectrum, the above calculation demonstrates *why* a theory with a logarithmic spectrum can arise from a Hamiltonian with a factorized representation of this form. The explanation rests in the series of partial supersymmetries which can be introduced into this problem. We are hopeful that this structure will provide sufficient added insight to constrain the potential $V_L(x)$, thus permitting some more exact statements about the logarithmic spectrum.

Perhaps even more strikingly, the Hamiltonian H_E in (3.10) acts on wavefunctions in R^∞, while the original Hamiltonian H_L acts on wavefunctions in R^1. These two Hamiltonians have the same spectra, and thus are thermodynamically equivalent. This physical equivalence of two theories, both bosonic, in such drastically different dimensions is striking. A fuller understanding of this phenomenon, and the factorization to which it is related, is under investigation. We note in passing a potential similarity to the sectorization phenomenon in [9]. We also note that, given the experience in field theory and string theory with phenomena such as bosonization, dualities and holography, finding alternative descriptions of the same underlying physics is highly valuable, and it would be instructive to establish an explicit mapping between H_L and H_E.

4. Asymptotic energy levels

In this section, we will see the broader utility of partial supersymmetry by using partial supersymmetry to obtain the asymptotic energy level spacing of quantum mechanical Hamiltonians in a general setting. While other methods exist to find these levels, the partial supersymmetry method introduced here is exceptionally easy from a calculational point of view, and obtains its effectiveness from the algebraic structure underlying the procedure. Again, as in the preceding section, we will see that partial supersymmetry can be used to obtain results about theories that are not intrinsically supersymmetric. At the end of this section, we use these asymptotic methods to return to the question of how the logarithmic spectrum of the previous section can be generated.

Suppose, then, that we have a theory—as before, non-relativistic quantum mechanics in one spatial dimension—with Hamiltonian $H(g)$ and with an energy spectrum given by $E(g, n), n = 1, 2, 3, \ldots$. Suppose further that we find a transformation $g \to g'$ such that the energy levels of the new Hamiltonian $H(g')$ satisfy $E(g', n) = E(g, 2n)$. In such a situation, $H(g)$ and $H(g')$ form a partially supersymmetric pair. Now suppose further that we find a temperature where the partition functions of both these Hamiltonians diverge, that is, a Hagedorn temperature. Then at such a temperature, we would see a signal of partial supersymmetry, namely that the ratio of the partition functions is $1/2$, as only the asymptotic density of states matters at a point where the partition functions diverge.

The key insight is that we can run this argument in reverse. Suppose we have a Hamiltonian $H(g)$ for which the partition function diverges at some temperature T_H (or inverse temperature β_H). Then given a transformation of the parameter $g \to g'$ such that at inverse temperature β_H the partition functions for $H(g)$ and $H(g')$ both diverge but satisfy

$$\frac{Z(g', \beta_H)}{Z(g, \beta_H)} = \frac{1}{2} \tag{4.1}$$

we can conclude that, asymptotically, the density of states of the second theory is half that of the first. This means that the asymptotic equality

$$E(g', n) \sim E(g, 2n) \tag{4.2}$$

holds, or in other words that, asymptotically, there is a partial supersymmetry. (To see that (4.2) follows from (4.1), we refer the reader to the appendix.) Now, in conjunction with some simple dimensional analysis, one can obtain the asymptotic level spacings in this theory.

What makes this an especially easy method for computing the asymptotic spacing of energy levels is that, due to the correspondence principle, the one necessary calculation—determination of the (Hagedorn) temperature at which the partition function diverges—can be performed in the classical limit, as it is the behaviour of the system at arbitrarily high energies that determines if and when the partition function diverges.

We present two examples to demonstrate this method, the second of which will connect us to the logarithmic spectrum of the previous section.

For the first example, consider the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + gx^r \tag{4.3}$$

where g and r are positive constants (and we can restrict to the positive x-axis if need be). We seek to find how the energy levels asymptotically depend on n. To find β_H, we calculate the classical partition function,

$$Z(g, m, \hbar, \beta) = \frac{1}{\hbar}\int \mathrm{d}p\, \mathrm{e}^{-\beta p^2/2m} \int \mathrm{d}x\, \mathrm{e}^{-\beta g x^r} \tag{4.4}$$

which, it is simple to see, yields

$$Z(g, m, \hbar, \beta) = \frac{1}{\hbar}\sqrt{\frac{2\pi m}{\beta}} \frac{c}{(\beta g)^{1/r}} \tag{4.5}$$

where c is a dimensionless constant. Consequently, the partition function diverges as $\beta_H \to 0$ or at infinite temperature.

It is easy to check that for $g' = 2^r g$, the partition functions satisfy

$$\frac{Z(g', m, \hbar, \beta_H)}{Z(g, m, \hbar, \beta_H)} = \frac{1}{2} \tag{4.6}$$

and, consequently, we have that

$$E(2^r g, \hbar, m, n) \sim E(g, \hbar, m, 2n) \tag{4.7}$$

where '$\sim$' denotes asymptotic equality. In other words, the combination of these two theories has partial supersymmetry in the asymptotic limit. Now it is a simple exercise in dimensional analysis to see that

$$E(g, \hbar, m, n) = \left(\frac{\hbar^2}{m}\right)^{\frac{r}{r+2}} g^{\frac{2}{r+2}} f(n) \tag{4.8}$$

for some function $f(n)$. Plugging the expression (4.8) into the partial supersymmetry relation (4.7), one readily determines that $f(n) = n^{2r/(r+2)}$, and thus that the asymptotic spacing of energy levels is given by

$$E(g, \hbar, m, n) \sim \left(\frac{\hbar^2}{m}\right)^{\frac{r}{r+2}} g^{\frac{2}{r+2}} n^{\frac{2r}{r+2}}. \tag{4.9}$$

One easily recognizes the familiar cases $r = 2$ (the harmonic oscillator) and $r \to \infty$ (the infinite square well).

Note that it is certainly possible to obtain this result through other methods, such as WKB or Bohr–Sommerfeld quantization, but the method presented here has the dual advantages of being far simpler in terms of the computations one must perform, and more instructive because it reveals the algebraic structure underlying the asymptotic limit. Indeed, it is the identification of a partial supersymmetry that holds in the asymptotic limit that is the key to both these advantages.

We now turn to a second calculation of asymptotic level spacing, and in doing so address the question of what potential $V_L(x)$ in (3.1) will lead to a logarithmic spectrum. Because of the nature of our techniques, it will only address this question asymptotically, but in doing so, will prove that a potential $V_L(x)$ exists.

Consider, then, a Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \epsilon_0 \ln(x). \tag{4.10}$$

First, we find the inverse temperature where the partition function for this Hamiltonian diverges. One finds

$$Z = \frac{1}{\hbar}\sqrt{\frac{2\pi m}{\beta}} \int_{x_0}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\beta\epsilon_0 \ln(x)} \tag{4.11}$$

which is finite for $\beta > 1/\epsilon_0$ but diverges at $\beta = 1/\epsilon_0$. (Note the need for an infrared cut-off x_0 for this potential.) Consequently, this theory has a true Hagedorn temperature, a finite temperature at which the partition function diverges, $T_H = 1/\beta_H = \epsilon_0$.

Now consider the transformation $V(x) \to V(x) + \epsilon_0 \ln(2)$. Under this transformation,

$$Z \to Z' = \frac{1}{2^{\beta\epsilon_0}} Z \tag{4.12}$$

at an arbitrary temperature, and thus at the Hagedorn temperature

$$\frac{Z'}{Z} = \frac{1}{2}. \tag{4.13}$$

Consequently, at this temperature, we have an asymptotic partial supersymmetry, and so $E'(n)$ and $E(2n)$ are asymptotically equal.

But the transformation on the potential has an utterly trivial effect on the energies: $E'(n) = E(n) + \epsilon_0 \ln(2)$. Putting all this together, we see that, asymptotically, $E(2n) \sim E(n) + \epsilon_0 \ln(2)$. Note that a similar argument could be made with a shift in the potential by $\epsilon_0 \ln(k)$ for any integer $k > 1$. As a result of this shift identity, one has that

$$E(n) \sim \epsilon_0 \ln(n). \tag{4.14}$$

Therefore, the logarithmic potential $\epsilon_0 \ln(x)$ produces, in the asymptotic regime, the logarithmic energy spectrum $\epsilon_0 \ln(n)$, the spectrum that we studied in the previous section.

We make two observations here. First, this result for the logarithmic potential can be verified by lengthier analytic methods. Second, we note that the asymptotic argument in this case is enough to ensure that there is some potential that will yield exactly the logarithmic spectrum. By modifying the small x behaviour of the potential in (4.10), one can get the lower energies to match up with the spectrum (3.2), while as long as the asymptotic behaviour of the potential at large x is unchanged, the spectrum will remain logarithmic at large n. Thus we have come full circle, using the partial supersymmetry analysis of asymptotic energy levels in quantum mechanics to shed light in turn on the theory with a logarithmic spectrum, the theory with which we started exploring the consequences of partial supersymmetry.

5. Conclusions

In this paper, I have demonstrated the usefulness of partial supersymmetry as a tool for the analysis of quantum mechanical systems. We have seen the kinds of energy spectra that are the hallmark of partial supersymmetry. We have also seen that we can enhance theories so that they have such a spectrum, use the partial supersymmetry of the enhanced theory to analyse the original theory and thus obtain results about theories that had no supersymmetry, partial or otherwise, to begin with.

Our two applications in this paper have been to understand the factorization into creation and annihilation operators that occurs when there is a purely logarithmic spectrum, and to obtain the asymptotic level spacings in non-relativistic quantum mechanics with only the simplest of calculations, since partial supersymmetry enables us to focus specifically on the asymptotic behaviour of the spectrum without distractions. Note that partial supersymmetry sheds light on the logarithmic case as well when using these asymptotic techniques.

There are two extensions under current investigation. One is the extension of these techniques to operators other than the Schrödinger operator of one-dimensional quantum mechanics. The other is the exploitation of algebraic aspects of partial supersymmetry at the operator level, not simply at the level of energy spectra.

In addition, it is important to note that there is an interpretation of this work that bears consideration. Note that both the section on the logarithmic spectrum and the section on the logarithmic potential (which asymptotically has a logarithmic spectrum) involve a common transformation, $H \to H + \epsilon_0 \ln(2)$.

In the first case, the shift in the energy leads us to thin out the energy levels, removing every other one (though of course we know, also, that this transformation gives the original theory at shifted energies). Thus it represents a kind of renormalization group transformation, where we map every two states into one. In the second case, the shift in the energy can be understood as a shift in the infrared cut-off (consider the partition function (4.11)). Again, a shift in the infrared cut-off should ultimately have no physical effect on the theory, and understanding how quantities are affected by this shift in the cut-off is the job of the renormalization group. Thus we see that there is an intimate connection between the applications of partial supersymmetry considered here and the renormalization group, and work is currently underway to understand better this connection and its interpretation.

Finally, should we have been surprised that a supersymmetric method can be used to obtain asymptotic level spacings? I would argue not. It is well known that supersymmetry can be used to obtain the energy levels in the known exactly soluble systems. The asymptotic limit of the quantum systems in question is exactly soluble (that is, the asymptotic behaviour can be determined exactly to leading order), and thus it is reasonable to expect that there would be a supersymmetric way to find this asymptotic behaviour. It turns out, however, that it is partial supersymmetry that is actually the relevant tool.

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Appendix A

In this appendix, I give a brief derivation to show how the relation (4.2) follows from (4.1).

To begin, we express the partition function in terms of the density of states. Define

$$\mathcal{Z}(g, \beta, \Lambda) = \int_{E_0}^{\Lambda} \rho(E, g)\, \mathrm{e}^{-\beta E} \tag{A.1}$$

where $\rho(E, g) = \sum_n \delta(E - E_n(g))$ is the density of states, E_0 is a lower bound for the energies in the spectrum and Λ is a high-energy cut-off. Then the actual partition function is

$$Z(g, \beta) = \lim_{\Lambda\to\infty} \mathcal{Z}(g, \beta, \Lambda). \tag{A.2}$$

For inverse temperature β_H, the partition functions $Z(g', \beta_H)$ and $Z(g, \beta_H)$ diverge, but have the finite ratio $1/2$. Thus for large finite Λ, we have the asymptotic expression

$$\frac{\mathcal{Z}(\beta_H, g', \Lambda)}{\mathcal{Z}(\beta_H, g, \Lambda)} \sim \frac{1}{2}. \tag{A.3}$$

Taking the derivative of this expression with respect to Λ yields

$$\rho(\Lambda, g')\mathcal{Z}(\beta_H, g, \Lambda) - \rho(\Lambda, g)\mathcal{Z}(\beta_H, g', \Lambda) \sim 0. \tag{A.4}$$

However, since we are working with this relationship in the limit of large Λ, we can insert the asymptotic relationship (A.3) into (A.4) . This enables us to conclude that, asymptotically in Λ, the density of states satisfies

$$\rho(g', \Lambda) \sim \tfrac{1}{2}\rho(g, \Lambda). \tag{A.5}$$

This means that, for large Λ, there are asymptotically twice as many states for $H(g)$ with energy below Λ than there are for $H(g')$, and thus that $E(g', n) \sim E(g, 2n)$.

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